Definable fiber bundles in an o-minimal expansion of a real closed field

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Abstract

Let $\mathcal{N} = (R, +\cdot, <, \dots)$ be an o-minimal expansion of the standard structure $(R, +\cdot, <)$ of a real closed field R. Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set X with fiber F and structure group K and $f, h : Y \to X$ definable maps between definable sets. We prove that if f and h are definably homotopic, then the induced definable fiber bundles $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.

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1 . Introduction.

Let $\mathcal{N} = (R, +\cdot, <, \dots)$ denote an o-minimal expansion of the standard structure $(R, +\cdot, <)$ of a real closed field R. The term "definable" means "definable with parameters in $\mathcal{N}$". General references on o-minimal structures are [4], [6], see also [23]. Further properties and constructions of them are studied in [5], [7]. J.P. Rolin, P. Speissegger and A.J. Wilkie [22] proved that there exist uncountably many o-minimal expansions of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. A definable category is a generalization of the semialgebraic category, and the definable category on $\mathcal{R}$ coincides the semialgebraic one.

The homotopy property of semialgebraic vector bundles is established in 12.7.7 [1] and it is studied in 2.10 [19] in the equivariant fiber bundle case. An equivariant version of 12.7.7 [1] is studied in [2], definable G sets and definable G maps are studied in [13], and definable C^rG manifolds and definable C^rG vector bundles are studied in [17], [15], [14].

In this paper, we use a definable space as in the sense of [4]. Every definable set is a definable space in this sense. Definable maps between definable spaces are assumed to be continuous and everything is considered in $\mathcal{N}$ unless otherwise stated.

Let X and Y be definable sets. Two definable maps $f, h : X \to Y$ are called *definably homotopic* if there exists a definable map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$.

Theorem 1.1. *Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set*

X with fiber F and structure group K. If two definable maps $f, h : Y \to X$ between definable sets are definably homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.

If $\mathcal{N}$ is an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of $\mathcal{R}$, then by 1.2 [13], if two definable maps between definable sets are homotopic, then they are definably homotopic. Hence we have the following corollary which is a generalization of 1.1 [16].

Corollary 1.2. *Let $\mathcal{N} = \mathcal{M}$ and let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set X with fiber F and structure group K. If two definable maps $f, h : Y \to X$ between definable sets are homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.*

We next consider definable G vector bundles and definable C^rG vector bundle when G is a finite group. *Strongly definable C^rG vector bundles* and *strongly definable G vector bundles* are defined similarly (See Definition 3.3).

Theorem 1.3. *Let G be a finite group, $\eta = (E, p, X)$ a definable G vector bundle over a definable G set X. Then η is strongly definable.*

Theorem 1.4. *Let G be a finite group, X an affine definable C^rG manifold X and $0 \le r < \infty$. For any two strongly definable C^rG vector bundles η and ζ over X, they are definably C^rG vector bundle isomorphic if and only if they are definably G vector bundle isomorphic. In particular a definable G vector bundle over X admits at most one strongly definable C^rG vector bundle structure.*

Let $\mathcal{N} = \mathcal{M}$ and K a compact subgroup of some $O_n(\mathbb{R})$. By the construction (19.6 [24]) of the n-universal principal bundle $\mathcal{B}_K = (B_K, p_K, X_K)$ relative to K, it is a Nash fiber bundle and E_K and X_K are compact affine Nash manifolds. Let F be an affine definable C^r manifold with an effective definable C^rK action and r a non-negative integer. Then by 2.5 [12], the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable C^r fiber bundle. A definable C^r fiber bundle $\eta = (E, p, X, F, K)$ is *strongly definable* if there exist some $\mathcal{B}_K = (B_K, p_K, X_K)$ and a definable C^r map $f : X \to X_K$ such that η is definably C^r fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. *Strongly definable fiber bundles* are defined similarly.

Theorem 1.5. *Let $\mathcal{N} = \mathcal{M}$, $\eta = (E, p, X, F, K)$ a strongly definable fiber bundle over a definable C^r manifold X such that K is a compact subgroup of some $O_n(\mathbb{R})$ and r a non-negative integer.*

(1) There exists a strongly definable C^r fiber bundle ζ over X such that ζ is definably fiber bundle isomorphic to η.

(2) If ζ' is another strongly definable C^r fiber bundle over X such that ζ' is definably fiber bundle isomorphic to η, then ζ' and ζ are definably C^r fiber bundle isomorphic.

In particular, (1) and (2) say that η admits a unique strongly definable C^r fiber bundle structure up to definable C^r fiber bundle isomorphism.

Theorem 1.5 is a generalization of 1.1 [12] and under the assumption of Theorem 1.5 K is a compact algebraic group (P133 [21]).

Theorem 1.6. *Let $\mathcal{M}$ be an o-minimal expansion of the exponential field $\mathbf{R}_{exp} = (\mathbb{R}, +, \cdot, <, e^x)$ admitting the C^∞ cell decomposition. Let $\eta = (E, p, X, F, K)$ be a strongly definable fiber bundle over a definable C^∞ manifold X such that K is a compact subgroup of some $O_n(\mathbb{R})$.*

(1) There exists a strongly definable C^∞ fiber bundle ζ over X such that ζ is definably fiber bundle isomorphic to η.

(2) If ζ' is another strongly definable C^∞ fiber bundle over X such that ζ' is definably fiber bundle isomorphic to η, then ζ' and ζ are definably C^∞ fiber bundle isomorphic.

In particular, (1) and (2) say that η admits a unique strongly definable C^∞ fiber bundle structure up to definable C^∞ fiber bundle isomorphism.

O. Le Gal and J.P. Rolin [20] proved that there exists some $\mathcal{M}$ which does not admit the C^∞ cell decomposition.

2 Definable fiber bundles

A *definable set* means a definable subset of some R^n. A group G is a *definable group* if G is a definable set such that the group operations $G \times G \to G$ and $G \to G$ are definable. A subgroup H of a definable group G is a *definable subgroup* if it is a definable subset of G. Every definable subgroup of a definable group is closed and a closed subgroup of a definable group is not necessarily definable. A group homomorphism (resp. An group isomorphism) between definable groups is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map. A *representation map* of a definable group G is a definable group homomorphism from G to some $O_n(R)$. A *representation* of G means some R^n with the linear action induced by a representation map $G \to O_n(R)$. In this paper, we assume that every representation of G is orthogonal. A G invariant definable subset of a representation of a definable group G is called a *definable G set*.

Let G be a definable group. A *definable set with a definable G action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \to X$ such that θ is a definable map. We simply write X instead of (X, θ). Clearly a definable G set is a definable set with a definable G action.

A *definable space* is an object obtained by pasting finitely many definable sets together along definable open subsets, and definable maps between definable spaces are defined similarly (see Chap. 10 [4]). Definable spaces are generalizations of semialgebraic spaces in the sense of [3].

Definition 2.1 ([15], [16]). (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) There exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

A bijective definable fiber bundle morphism $\overline{f} : E \to E'$ is called a *definable fiber bundle equivalence* if f is a definable homeomorphism and $(\overline{f})^{-1} : E'$

$\to E$ is a definable fiber bundle morphism covering $f^{-1} : X' \to X$. A definable fiber bundle equivalence is a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$. We say that η is *definably trivial* if η is definably fiber bundle isomorphic to the trivial bundle $(X \times F, proj, X, F, K)$, where $proj : X \times F \to X$ denotes the projection onto the first factor.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K.

Proposition 2.2 (3.2 [8]). *For any finite covering of a definable set $X \times [0,1]$ by definable open sets $\{V_j\}_{j=1}^p$, there exists a finite cover of X by definable open sets $\{U_i\}_{i=1}^q$ and, for each i definable functions $0 = \phi_{i,0} < \cdots < \phi_{i,k_i} = 1$ from U_i to R such that for each pair (i, l) with $1 \le i \le q$ and $0 \le l < k_i$ there exists some j such that $\{(x, y)|x \in U_i, \phi_{i,l}(x) \le y \le \phi_{i,l+1}(x)\} \subset V_j$.*

By a way similar of the proof of 2.2 [16], we have the following lemma.

Lemma 2.3. *Let A be a definable set, $X_1 = \{(x_1, x_2) \in A \times [0,1]|f_1(x_1) < x_2 \le f_2(x_1)\}, X_2 = \{(x_1, x_2) \in A \times [0,1]|f_2(x_1) \le x_2 < f_3(x_1)\}$ and $\eta = (E, p, X, F, K)$ a definable fiber bundle over $X = X_1 \cup X_2$, where $f_i : A \to [0,1]$, $(1 \le i \le 3)$, are definable functions with $f_1 < f_2 < f_3$. If $\eta|X_1$ and $\eta|X_2$ are definably trivial, then η is definably trivial.*

The above two results proves the following proposition.

Proposition 2.4. *Let X be a definable set and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a finite definable open covering $\{U_i\}_i$ of X such that each $\eta|(U_i \times [0,1])$ is definable trivial.*

Theorem 2.5. *Let X be a definable set, $r : X \times [0,1] \to X \times [0,1], r(x,t) = (x,1)$ and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a definable fiber bundle morphism $\phi : E \to E$ with $p \circ \phi = r \circ p$.*

We need a definable partition of unity to prove Theorem 2.5.

Proposition 2.6 (6.3.7 [4]). *Let X be a definable subset of R^n and $\{U_i\}_{i=1}^l$ a finite definable open covering of X. Then there exist definable functions $\lambda_1, \dots, \lambda_l : X \to R$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

We call $\{\lambda_i\}$ in Proposition 2.6 a *definable partition of unity subordinate to* $\{U_i\}$. In Proposition 2.6, we can replace $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$ by $\max_{1 \le i \le n} f_i(x) = 1$ for any $x \in X$.

Proof of Theorem 2.5. By Proposition 2.4, we can find a finite definable open covering $\{U_i \times [0,1]\}_{i=1}^n$ of $X \times [0,1]$ such that each $\eta|(U_i \times [0,1])$ is definably trivial. Since $\{U_i\}_{i=1}^n$ is a definable open covering of X and by Proposition 2.6, there exist definable functions $f_1, \dots, f_n : X \to [0,1]$ such that:

1. The support of each f_i is contained in U_i.
2. $\max_{1 \le i \le n} f_i(x) = 1$ for any $x \in X$.

Let $\{h_i : U_i \times [0,1] \times F \to p^{-1}(U_i \times [0,1])\}_{i=1}^n$ be definable local trivializations of η. Define

$$(u_i, r_i) : (E, X \times [0,1]) \to (E, X \times [0,1]),$$

$$r_i(x,t) = \begin{cases} (x, \max(f_i(x), t)), & (x,t) \in U_i \times [0,1] \\ (x,t), & \text{otherwise} \end{cases},$$

$u_i(h_i(x,t,y)) = h_i(x, \max(f_i(x),t), y),$
for any $(x,t,y) \in U_i \times [0,1] \times F$,
u_i is the identity outside $p^{-1}(U_i \times [0,1])$. Then $r = r_n \circ \cdots \circ r_1$. Therefore $\phi := u_n \circ \cdots \circ u_1 : E \to E$ is the required definable fiber bundle morphism. □

By Theorem 2.5, we have Theorem 1.1. □

A definable set X is *definably contractible* if there exist some point $x_0 \in X$ and a definable map $F : X \times [0,1] \to X$ such that for any $x \in X, F(x,0) = x_0$ and $F(x,1) = x$.

Corollary 2.7. *Every definable fiber bundle over a definably contractible definable set is definably trivial.*

3 Definable C^rG fiber bundles and definable C^rG vector bundles

In $\mathcal{N}$, we can define definable C^r manifolds, definable C^rG manifolds and affine definable C^rG manifolds as well as $\mathcal{M}$ [15].

Definition 3.1 ([15])**.** Let G be a definable C^r group and $0 \le r \le \infty$.

(1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable C^r fiber bundle* if the total space E and the base space X are definable C^r manifolds, the structure group K is a definable C^r group, the fiber F is a definable C^rK manifold with an effective action, the projection p is a definable C^r map and all transition functions of η are definable C^r maps. A *principal definable C^r fiber bundle* is defined similarly.

(2) *Definable C^r fiber bundle morphisms*, *definable C^r fiber bundle equivalences*, *definable C^r fiber bundle isomorphisms* between definable C^r fiber bundles and *definable C^r sections* of a definable C^r fiber bundle are defined similarly.

(3) A definable fiber bundle (E, p, X, F, K) (respectively A principal definable fiber bundle (E, p, X, K)) is called a *definable G fiber bundle* (respectively a *principal definable G fiber bundle*) if the total space E is a definable G space such that G acts on E through definable fiber bundle equivalences, the base space X is a definable set with a definable G action and the projection p is a definable G map. A *definable C^rG fiber bundle* and *principal definable C^rG fiber bundle* are defined similarly.

(4) A definable fiber bundle morphism (respectively A definable fiber bundle equivalence, A definable fiber bundle isomorphism) between definable G fiber bundles is a *definable G fiber bundle morphism* (respectively a *definable G fiber bundle equivalence*, a *definable G fiber bundle isomorphism*) if it is a G map.

(5) A *definable G section* of a definable G fiber bundle means a definable section which is a G map.

(6) A *definable C^r vector bundle* (resp. A *definable C^rG vector bundle*) is a definable C^r fiber bundle (resp. a definable C^rG fiber bundle) with fiber R^n and structure group $GL_n(R)$.

(7) Let $\eta_1 = (E, p, X)$ and $\eta_2 = (E', p', X)$ be definable C^rG vector bundles over X. A *definable C^rG vector bundle morphism* $\eta_1 \to \eta_2$ is a definable C^rG map $f : E \to E'$ such that $p = p' \circ f$ and f is linear on each fiber. A definable C^rG vector bundle morphism $h : \eta \to \eta'$ is a *definable C^rG vector bundle isomorphism* if there exists a definable C^rG vector bundle morphism $\overline{h} : \eta' \to \eta$ such that $h \circ \overline{h} = id$ and $\overline{h} \circ h = id$.

Definition 3.2. Let G be either a definable group or a definable C^r group and $0 \le r \le \infty$. Let Ω be an n-dimensional

representation of G and let B be the representation map $G \to O_n(R)$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:
$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\}$, $E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\}$, $u : E(\Omega, k) \to G(\Omega, k) : u((A, v)) = A$, where A' denotes the transposed matrix of A and $Tr\ A$ stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k*. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets.

Definition 3.3. (1) Let G be a definable group. A definable G vector bundle $\eta = (E, p, X)$ over a definable G set X is called *strongly definable* if there exist a representation Ω of G and a definable G map $f : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

(2) Let G be a definable C^r group and $0 \le r \le \infty$. A definable C^rG vector bundle $\eta = (E, p, X)$ over an affine definable C^rG manifold X is called *strongly definable* if there exist a representation Ω of G and a definable C^rG map $f : X \to G(\Omega, k)$ such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

We can define the Whitney sum of two definable G vector bundles and that of two definable C^rG vector bundles in a way similar to P256 [1] and P85 Example 3 [18].

By a way similar to the proof of 3.6 [2], we have the following theorem.

Theorem 3.4. *Let G be a finite group and $\eta = (E, p, X)$ a definable G vector bundle of rank k over a definable G set X. Then the following five properties are equivalent.*

(1) η is strongly definable.

(2) There exists a surjective definable G vector bundle morphism from a trivial G vector bundle $\underline{\Omega} = X \times \Omega$ onto η for some representation space Ω of G.

(3) There exists an injective definable G vector bundle morphism from η to a trivial G vector bundle $\underline{\Xi} = X \times \Xi$ for some representation space Ξ of G.

(4) There exist non-equivariant definable global sections $s_1, \dots, s_n : X \to E$ such that:

(1) The vectors $s_1(x), \dots, s_n(x)$ generate the fiber $E|_x = p^{-1}(x)$ for all $x \in X$.

(2) The sections $s_1, \dots, s_n$ generate a finite dimensional G invariant vector subspace of $\Gamma(E)$, where $\Gamma(E)$ denotes the set of all continuous global sections of E with the natural G action, namely $(g \cdot s)(x) = g(s(g^{-1}(x)))$ for all $g \in G$ and $x \in X$.

(5) There exists a definable vector bundle η over X such that $\xi \oplus \eta$ is definably G vector bundle isomorphic to a trivial G vector bundle $\underline{\Omega'} = X \times \Omega'$ for some G representation space Ω'.

Theorem 3.5 ([9])**.** *Let X, Y be definable C^r submanifolds of R^n, R^m, respectively and $0 \le s < r < \infty$. Every definable C^s map $f : X \to Y$ is approximated by a definable C^r map with respect to the definable C^s topology.*

To consider an equivariant version of Theorem 3.5, we need the averaging function.

Let $G = \{g_1, \dots, g_m\}$, X an affine definable C^rG manifold and Ω a representation of G. Then we define the averaging function $A : C^r(X, \Omega) \to C^r(X, \Omega)$ by $A(f)(x) = \frac{1}{m} \sum_{i=1}^m g_i^{-1} f(g_i x)$.

Then we have the following proposition.

Proposition 3.6. *(1) If f is a definable C^r map, then $A(f)$ is a definable C^rG map.*

(2) Let $Def^r(X,\Omega)$ (resp. $Def^r_G(X,\Omega)$) denote the set of definable C^r maps (resp. definable C^rG maps) from X to Ω. Then $A|Def^r_G(X,\Omega) = id_{Def^r_G(X,\Omega)}$ and $A(Def^r(X,\Omega)) = Def^r_G(X,\Omega)$.

(3) $A : Def^r(X,\Omega) \to Def^r(X,\Omega)$ is continuous in the definable C^r topology.

As in the proof of 1.2 [14], we have the following result.

Proposition 3.7. *Let G be a finite group, X a definable C^rG submanifold of a representation Ω of G and $1 \le r < \infty$. Then there exists a definable C^rG tubular neighborhood (U,θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable C^rG map such that $\theta|X = id_X$.*

By Theorem 3.5, Proposition 3.6 and 3.7, we have the following theorem.

Theorem 3.8. *Let G be a finite group and X, Y definable C^rG submanifolds of representations Ω, Ξ of G, respectively and $0 \le s < r < \infty$. Every definable C^sG map $f : X \to Y$ is approximated by a definable C^rG map with respect to the definable C^s topology.*

Proof of Theorem 1.3. It suffices to construct a finite family of definable sections of η as in Theorem 3.4. By the definition of definable vector bundles, η has a finite family of definable local trivializations $\{U_i, \phi_i : U_i \times R^k \to p^{-1}(U_i)\}_{i=1}^l$, where k denotes the rank of η. By Proposition 2.6, we can find a definable partition of unity $\{\lambda_i\}$ subordinate to $\{U_i\}$. Thus we have global definable sections

$$s_{ij}(x) = \begin{cases} \phi_i(x, \lambda_i(x)e_j), & x \in U_i \\ \text{the zero section}, & otherwise \end{cases},$$

$1 \le i \le l, 1 \le j \le k$, where e_j denotes the j-th fundamental vector of R^k. Since G is finite, we obtain the required finite family of sections $\{g \cdot s_{ij} | 1 \le i \le l, 1 \le j \le k, g \in G\}$. □

Proof of Theorem 1.4. Since η and ζ are strongly definable C^rG vector bundles, as in the proof of 3.1 [11], $Hom(\eta,\zeta)$ is a strongly definable C^rG vector bundle. Since η and ζ are definably G vector bundle isomorphic, this definable G vector bundle isomorphism defines a definable G section of $Hom(\eta,\zeta)$. Using Theorem 3.8, by a way similar to [11], η and ζ are definably C^rG vector bundle isomorphic. □

Proposition 3.9 (2.11 [12]). *Let X_1, X_2 be definable C^r manifolds, η_1, η_2 principal definable C^r fiber bundles over X_1, X_2 whose structure groups are a compact subgroup K of some $O(n)$, respectively. Let $f_1 : X_1 \to X_2$ be a definable C^r map.*

(1) There exists a definable C^r fiber bundle (η_1, η_2, f) over X_1 such that its definable C^r sections are in bijective correspondence with the definable C^r fiber bundle morphisms $\eta_1 \to \eta_2$ covering f.

(2) If η_1, η_2 are strongly definable, then (η_1, η_2, f) is strongly definable.

Proof of Theorem 1.5. (1) Since η is strongly definable, we can find the n-universal bundle $\mathcal{B}_K$ and a definable map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to η. By Theorem 3.5, we have a definable C^r map $h : X \to X_K$ as an approximation of f. In particular h is definably homotopic to f. Thus by Theorem 1.1, $\zeta := h^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$ and ζ is a strongly definable C^r fiber bundle.

(2) Let ζ' be another strongly definable C^r fiber bundle over X such that ζ' is definably fiber bundle isomorphic to η. Consider the strongly definable C^r fiber bundle (ζ, ζ', id_X) whose sections represent the fiber bundle isomorphisms between ζ and ζ' which is defined in Proposition 3.9. Then it has a continuous section. By a way similar to the proof of 2.12 [12], it admits a definable C^r section. This section gives a definable C^r fiber bundle isomorphism between ζ and ζ'. □

Theorem 3.10 ([10]). *Let $\mathcal{M}$ be an o-minimal expansion of the exponential field*

$\mathbf{R}_{exp} = (\mathbb{R}, +, \cdot, <, e^x)$ admitting the C^∞ cell decomposition and $0 \le r < \infty$. Then every definable C^r map between definable C^∞ manifolds is approximated by a definable C^∞ map in the definable C^r topology.

Using Theorem 3.10, a similar proof of Theorem 1.5 proves Theorem 1.6. □

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